WESLEY COLLEGE
By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER ONE 2018
TEST 1: Complex Numbers

Name: — Solutions —

Wednesday 7 March

Time: 55 minutes **Mark** /46= %

- Answer all questions neatly in the spaces provided. **Show all working.**
- You are permitted to use the Formula Sheet in **both** sections of the test.
- You are permitted one A4 page (one side) of notes in the calculator assumed section.

Calculator free section **Suggested time: 25 minutes** /22

Question 1 (6 marks)

$$f(z) = z^3 - 5z^2 + 17z - 13$$

a) Show $(z-1)$ is a factor of $f(z)$

$f(1) = (1)^3 - 5(1)^2 + 17(1) - 13 = 1 - 5 + 17 - 13$
$= 0 \quad \therefore \quad (z-1)$ is a factor

[1]

b) Re-write $f(z) = (z-1)Q(z) + R$

$$\begin{array}{c|cccc} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & -13 \\ \hline & 1 & -4 & 13 & 0 \end{array}$$

$f(z) = (z-1)(z^2 - 4z + 13) + 0$ ✓ ✓

[2]

c) Hence find all the roots of the equation, giving your answers in the form $a + ib$ where a and b are integers.

$(z-1)(z^2 - 4z + 13) = 0$

$(z-1)\left[(z-2)^2 - 4 + 13\right] = 0$

$(z-1)\left[(z-2)^2 + 9\right] = 0$ ✓

$z = 1$ ✓ $\quad z = 2 \pm 3i$ ✓

[3]

Question 2 (9 marks)

The complex numbers z_1 z_2 and z_3 are given by

$$z_1 = 7 - i \qquad z_2 = 1 + i\sqrt{3} \qquad z_3 = a + ib$$

where a and b are real constants.

a) Given $|z_1 z_3| = 50$, find $|z_3|$

$|z_1| = \sqrt{50} = 5\sqrt{2}$ ✓

$\therefore |z_3| = \frac{50}{5\sqrt{2}} = \frac{10}{\sqrt{2}} = 5\sqrt{2}$ ✓

[2]

b) Given also $\arg\left(\frac{z_2}{z_3}\right) = \frac{7\pi}{12}$, find $\arg(z_3)$

$\arg(z_2) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}.$ ✓

$\therefore \arg(z_3) = \frac{\pi}{3} - \frac{7\pi}{12} = -\frac{3\pi}{12} = -\frac{\pi}{4}$ ✓

[2]

c) Determine the values of a and b.

$z_3 = 5\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$= 5 - 5i$

$a = 5$ ✓

$b = -5.$ ✓

[2]

d) Show that $\frac{z_1}{z_3} = \frac{1}{5}(4 + 3i)$

$\frac{z_1}{z_3} = \frac{(7-i)}{(5-5i)} \times \frac{(5+5i)}{(5+5i)}$ ✓ $= \frac{35 - 5i^2 + 35i - 5i}{25+25}$

$= \frac{40 + 30i}{50}$ ✓

$= \frac{1}{5}(4 + 3i)$ shown ✓

[3]

Question 3 (3 marks)

Given $w = a + ib$, find the values of a and b when $2w - 3\overline{w} = 3 - 20i$.

$2(a+ib) - 3(a-ib) = 3-20i$ ✓

Re: $2a - 3a = 3 \Rightarrow a = -3$ ✓

Im: $2b + 3b = -20 \Rightarrow b = -4$ ✓

[3]

Question 4 (4 marks)

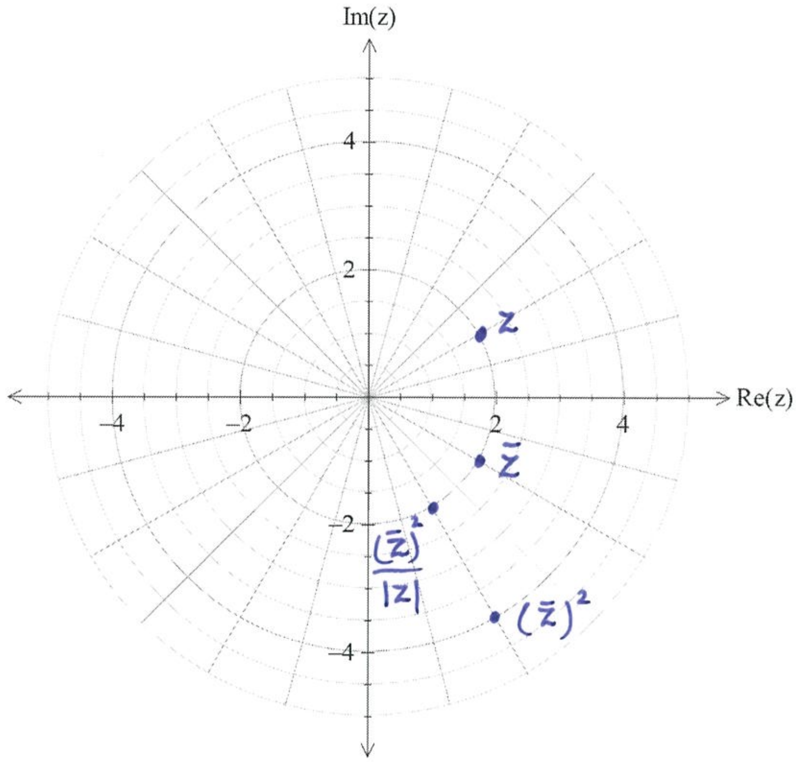
If $z = 2cis\left(\dfrac{\pi}{6}\right)$, illustrate, and label, on an Argand diagram the points:

i. z

ii. $\overline{z}$

iii. $\left(\overline{z}\right)^2$

iv. $\dfrac{\left(\overline{z}\right)^2}{|z|}$



[4]

✓1 each!

Name: ______________________

Calculator assumed section **Suggested time: 25 minutes** **/24**

Question 5 (5 marks)

Illustrate the solutions to the equation $z^3 = -4 + 4\sqrt{3}i$.

$|-4+4\sqrt{3}i| = 8.$

$\arg(-4+4\sqrt{3}i) = \frac{2\pi}{3}.$ ✓

$z^3 = 8 \operatorname{cis}\left(\frac{2\pi}{3}\right) = (r \operatorname{cis}\theta)^3$

$= r^3 \operatorname{cis} 3\theta.$

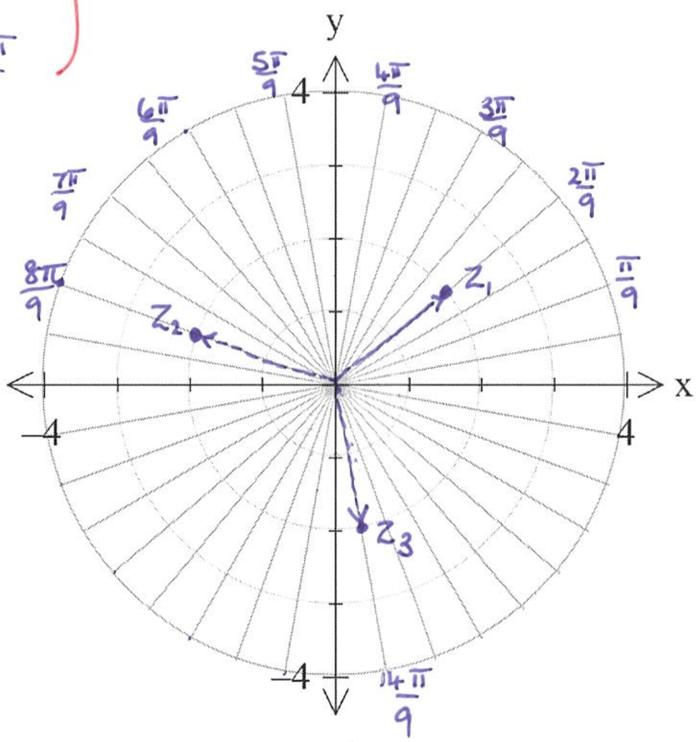
$r^3 = 8 \Rightarrow r = 2$

$3\theta = \frac{2\pi}{3} + 2n\pi. \quad \therefore \theta = \frac{\left[\frac{2\pi}{3} + 2n\pi\right]}{3}$ ✓

$n=0 \quad \theta = \frac{2\pi}{9}$

$n=1 \quad \theta = \frac{8\pi}{9}$ ✓✓

$n=2 \quad \theta = \frac{14\pi}{9}$



✓✓

[6]

$z_1 = 2 \operatorname{cis} \frac{2\pi}{9}$

$z_2 = 2 \operatorname{cis} \frac{8\pi}{9}$

$z_3 = 2 \operatorname{cis} \frac{14\pi}{9}$

Question 6 (5 marks)

The complex number z is given by $z = \dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}$.

a) State z in the form $\lambda\left(1-i\sqrt{3}\right)$, where λ is rational number you should find.

$$z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = -\frac{1}{2}\left(1-\sqrt{3}i\right)$$

[1]

b) State the modulus and argument of z.

$|z| = 1$ ✓ $\quad \arg(z) = \frac{2\pi}{3}$ ✓

[2]

c) Hence, or otherwise, find the modulus and argument of $\left(\dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^4$

$$z^4 = \left[1 \operatorname{cis} \frac{2\pi}{3}\right]^4 = 1^4 \times \operatorname{cis} \frac{2\pi}{3} \times 4$$ ✓

modulus $= 1$ ✓

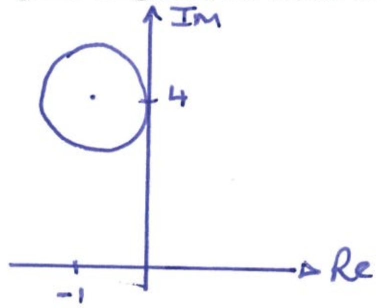
argument $= \frac{8\pi}{3} = \frac{2\pi}{3}$.

[2]

Question 7 (5 marks)

Given that $|z+1-4i| = 1$ $\quad |z-(-1+4i)| = 1.$

a) sketch, in an Argand diagram, the locus of z.

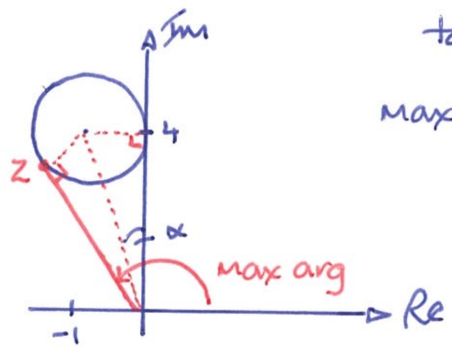


centre $(-1, 4)$ ✓

radius $1.$ ✓

[2]

b) find the maximum value of arg(z) in degrees to one decimal place.



$\tan\alpha = \frac{1}{4} \Rightarrow \alpha = 14.04$ ✓

Max arg $= 90 + 2\alpha$ ✓

$= 90 + 28.08$

$= 118.08$

$\simeq 118.1°$. ✓

[3]

Question 8 (9 marks)

The complex number $w = x + iy$, where x and y are real, satisfies the equation

$$|w + 1 + 8i| = 3|w + 1|.$$

The complex number w is represented by the point P in the Argand diagram.

a) Show that the locus of P is a circle and state the centre and radius of this circle.

$$|x + iy + 1 + 8i| = 3|x + iy + 1|$$

$$(x+1)^2 + (y+8)^2 = 9[(x+1)^2 + y^2]$$

$$\therefore \quad 8(x+1)^2 + 9y^2 - (y+8)^2 = 0$$

$$8(x+1)^2 + 8y^2 - 16y - 64 = 0$$

$$8(x+1)^2 + 8(y^2 - 2y - 8) = 0$$

$$(x+1)^2 + (y-1)^2 - 9 = 0$$

$$(x+1)^2 + (y-1)^2 = 3^2$$

centre at $(-1 + i)$ radius $= 3$.

[3]

Question contd on next page ...

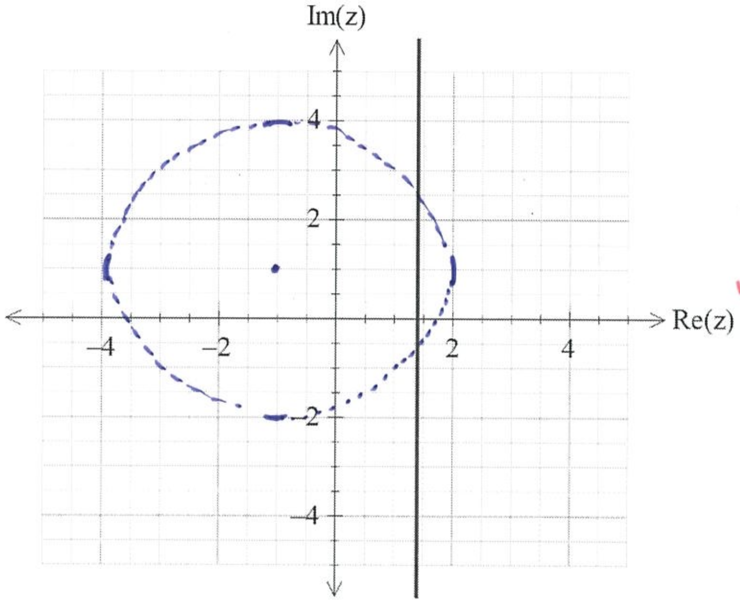
b) The locus $|w| = \left|w - \frac{14}{5}\right|$ is represented on the Argand diagram below.

i. Explain why the locus is as shown.

It is the perpendicular bisector of the line joining $(0,0)$ to $\left(\frac{14}{5}, 0\right)$

[2]

ii. Add the locus of P to the diagram.



✓ shape
✓ centre/radius

[2]

c) Find the complex numbers corresponding to the points of intersection of these loci, giving your answers in the form $a + ib,\ a, b \in \mathbb{R}$.

Sub $x = \frac{7}{5}$ into locus of P.

$$\left(\frac{7}{5} + 1\right)^2 + (y-1)^2 = 9$$

$$(y-1)^2 = 9 - \frac{144}{25}$$

$$\therefore y = \pm\sqrt{\frac{81}{25}} + 1$$

$$= \pm\frac{9}{5} + 1$$

[2]

Two points are $\frac{7}{5} - \frac{4}{5}i$ & $\frac{7}{5} + \frac{14}{5}i$